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B.Sc. part-I (H), paper-I Date-20-04-2020

Set Theory

Q.1. Define cardinality of a set and cardinal numbers. Construct an arithmetic of cardinal numbers. Point out the distinction between the cardinal arithmetic and the arithmetic of the natural numbers.

Soln. Cardinality of a set: →

Let X and Y are two sets and $f: X \to Y$ be a function which is one-one and onto then we call X and Y equipotent or of the same power or equinumerous or cardinally equivalent. We also say that X and Y are sets having equal or same cardinal or are sets of the same or equal cardinality. In such a case we express this by writing $X \approx Y$.

Cardinal numbers: → If x is an ordinal and for each equinumerous ordinal α, $x \leq \alpha$ then we call x a Cardinal Number.

Let λ, μ are two cardinals and X, Y are two sets where $\text{Card } X = \lambda$, $\text{Card } Y = \mu$ and $X \cap Y = \phi$. We define

$$\lambda + \mu = \text{card}(X \cup Y)$$

We call $\lambda + \mu$ the sum of λ and μ.

Here it is possible to take two disjoint sets. In fact, if X_1 and Y_1 are two sets and $\text{card } X_1 = \lambda$, $\text{card } Y_1 = \mu$ then by taking

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$X = X_1 \times \{0\}, \; Y = Y_1 \times \{1\}$

We get $X \cap Y = \phi$

Every natural number n is an ordinal. Also, for each equinumerous ordinal α (i.e. $n \approx \alpha$) we have, $n = \alpha$, since n being finite, α is also finite and at the same time $n \neq \alpha$ is not possible.

Q.2. State and prove well-ordering theorem and deduce axiom of choice from well-ordering theorem.

Soln: Statement: → A partially ordered set X is said to be well ordered if every subset of X contains a first element.

For example; the set N of natural numbers with the natural order is well-ordered since every subset of N has a first element.

proof: Let X be any set and non-empty. Since, the empty set ϕ can obviously be looked upon as a well ordered set. We have to find that a well-ordering on X. Let F be the family of all well-ordered subset of X. Then an element $W \in F$ is a pair $W = (A, \leq)$ where $A \subseteq X$ and $\leq$ defines a well-ordering in A. If a set $A \subseteq X$ can be well-ordered in several ways then A with each of these well-orderings appears as a distinct element in F. The family F is obviously non-empty since every singlet $\{x\}$ with $x \in X$ can be looked upon as a well-ordered subset of X.

Let X be a non-empty set the members of which are non-empty sets disjoint two by two, then a set C can be obtained such that its formation is done by taking one and only one element out of each member of X.

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B.Sc-part II(H), paper III

Differential Calculus

Q.1. If $u = \log(x^2+y^2)$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Soln: We have, $u = \log(x^2+y^2)$

Differentiating partially w.r.t. x,

we get $\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \times 2x = \frac{2x}{x^2+y^2}$

Again differentiating partially w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{2[1(x^2+y^2) - 2x \cdot x]}{(x^2+y^2)^2} = \frac{2[x^2+y^2-2x^2]}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2} \quad \text{——— I}$$

Similarly, diff. partially w.r.t. y, we get

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$$

Again, differentiating partially w.r.t. y, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{2[1(x^2+y^2) - 2y \cdot y]}{(x^2+y^2)^2} = \frac{2[x^2+y^2-2y^2]}{(x^2+y^2)^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2} \quad \text{——— II}$$

Adding I and II

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2} = \frac{2y^2-2x^2+2x^2-2y^2}{(x^2+y^2)^2}$$

$$= 0$$

Proved

B.Sc. part-II (U), paper-III

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Q.2.. If $u = t^{-\frac{3}{2}} e^{-\frac{r^2}{4\kappa t}}$, prove that

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\cdot\frac{\partial u}{\partial r}\right)$$

Soln: We have, $u = t^{-3/2} \cdot e^{-\frac{r^2}{4\kappa t}}$ —— (I)

Differentiating (I) partially w.r.t. t, we get keeping r constants, we get

$$\frac{\partial u}{\partial t} = -\frac{3}{2} t^{-\frac{5}{2}} \cdot e^{-\frac{r^2}{4\kappa t}} + t^{-\frac{3}{2}} e^{-\frac{r^2}{4\kappa t}}\left(-\frac{r^2}{4K}\right)\left(-\frac{1}{t^2}\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} = -\frac{3}{2} t^{-\frac{5}{2}} e^{-\frac{r^2}{4\kappa t}} + \frac{r^2}{4K} t^{-\frac{7}{2}} e^{-\frac{r^2}{4\kappa t}} \quad \text{—— (II)}$$

Differentiating I partially w.r. to r (keeping t as constant), we get

$$\frac{\partial u}{\partial r} = t^{-\frac{3}{2}} e^{-\frac{r^2}{4Kt}}\left(-\frac{2r}{4Kt}\right)$$

$$\Rightarrow \frac{\partial u}{\partial r} = -\frac{t^{-\frac{5}{2}}}{2K}\left(r e^{-\frac{r^2}{4Kt}}\right) \quad \text{—— (III)}$$

Differentiating III w.r. to r (keeping t as constant), we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{t^{-\frac{5}{2}}}{2K}\left(e^{-\frac{r^2}{4\kappa t}} - \frac{2r^2}{4Kt}\right) \quad \text{—— (IV)}$$

$$R.H.S = K\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) \qquad -t^{-\frac{5}{2}} e^{-\frac{r^2}{4Kt}}$$

$$= -\frac{1}{2} t^{-\frac{3}{2}} e^{-\frac{r^2}{4Kt}} + \frac{r^2}{4K} t^{-\frac{7}{2}} e^{-\frac{r^2}{4Kt}}$$

$$= -\frac{3}{2} t^{-\frac{5}{2}} e^{-\frac{r^2}{4Kt}} + \frac{r^2}{4K} t^{-\frac{7}{2}} e^{-\frac{r^2}{4Kt}}$$

$$= \frac{\partial u}{\partial t} = L.H.S \qquad \textbf{Proved.}$$

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Differential Calculus

Q.2. If $y = f(x+ct) + \phi(x-ct)$, show that

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Soln: We have $y = f(x+ct) + \phi(x-ct)$ — (I)

Differentiating partially with respect to t, we get

$$\frac{\partial y}{\partial t} = f'(x+ct).c + \phi'(x-ct).(-c)$$

Again, differentiating this partially w.r.to. t, we get

$$\frac{\partial^2 y}{\partial t^2} = f''(x+ct)c^2 + \phi''(x-ct)(-c)^2$$

$$= c^2[f''(x+ct) + \phi''(x-ct)] \text{ — II}$$

Differentiating (I) partially with respect to x, we get

$$\frac{\partial y}{\partial x} = f'(x+ct).1 + \phi'(x-ct).1$$

Again, differentiating this partially with respect to x, we get

$$\frac{\partial^2 y}{\partial x^2} = f''(x+ct) + \phi''(x-ct) \text{ — (III)}$$

From II and III, we find that

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

Proved

Differential Calculus

Q.4. If ∇^2 stands for the operator.

$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ prove that $\nabla^2 r=0$.

Where $\frac{1}{r^2}=x^2+y^2+z^2$

Soln. We have, $\frac{1}{r^2}=x^2+y^2+z^2$ ——— (I)

Differentiating (I) partially w.r. to x (keeping y and z as constants), we get

$$-\frac{2}{r^3}\frac{\partial r}{\partial x}=2x \Rightarrow \frac{1}{r^2}\frac{\partial r}{\partial x}=-x \quad \text{——— (II)}$$

Again, differentiating (II) partially w.r. to x (keeping y and z as constants) we get

$$-\frac{3}{r^4}\left(\frac{\partial r}{\partial x}\right)^2+\frac{1}{r^3}\cdot\frac{\partial^2 r}{\partial x^2}=-1$$

$$\Rightarrow -\frac{3}{r^4}(-xr^2)^2+\frac{1}{r^3}\frac{\partial^2 r}{\partial x^2}=-1 \quad [\text{from II}]$$

$$\Rightarrow -3x^2r^2+\frac{1}{r^3}\frac{\partial^2 r}{\partial x^2}=-1 \quad \text{——— (III)}$$

Similarly, $-3y^2r^2+\frac{1}{r^3}\frac{\partial^2 r}{\partial y^2}=-1$ ——— (IV)

and $-3z^2r^2+\frac{1}{r^3}\frac{\partial^2 r}{\partial z^2}=-1$ ——— (V)

Adding (III), (IV) and (V), we get

$$-3r^2(x^2+y^2+z^2)+\frac{1}{r^3}\left(\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 r}{\partial z^2}\right)=-3$$

$$\Rightarrow -3\cancel{r^2}\cdot\frac{1}{\cancel{r^2}}+\frac{1}{r^3}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)r=-3 \quad [\text{from I}]$$

$$\Rightarrow \cancel{-3}+\frac{1}{r^3}\nabla^2 r=\cancel{-3}$$

Hence, $\nabla^2 r=0$ proved